Blowing up (surfaces)

Def Y scheme, $Z \subseteq Y$ closed subscheme

blow up of Y at Z

$$Bl_Z(Y) = \mathrm{Proj}_Y \left(\bigoplus_{n \geq 0} \mathcal{I}_Z^n t^n \right) \xrightarrow{\beta} Y$$

Example $Y = \mathbb{A}^2$, $Z = (0,0) \in \mathbb{A}^2$, $I_Z = (x,y) \subseteq k[x,y]$

$$\varphi : k[x,y][\tilde{x},\tilde{y}] \twoheadrightarrow \bigoplus_{n \geq 0} I_Z^n \qquad \text{morphism of graded } k[x,y]\text{-algebras}$$

$$\tilde{x} \longmapsto xt \in I_Z^1$$
$$\tilde{y} \longmapsto yt \in I_Z^1$$

Lemma $\ker(\varphi) = (x\tilde{y} - y\tilde{x})$

$$Bl_0 \mathbb{A}^2 = \mathrm{Proj}_{k[x,y]} \left(k[x,y][\tilde{x},\tilde{y}] / (x\tilde{y} - y\tilde{x}) \right)$$
$$= \{ (x,y)[\tilde{x}:\tilde{y}] \mid x\tilde{y} - y\tilde{x} \} \subseteq \mathbb{A}^2 \times \mathbb{P}^1$$

Union of two affine charts $U_x \cup U_y = Bl_0 \mathbb{A}^2$

$$U_x = (\mathbb{A}^2 \times D_{\tilde{x}}) \cap Bl_0(\mathbb{A}^2) = \left\{ (x,y)[\tilde{x}:\tilde{y}] \;\middle|\; \begin{matrix} x\tilde{y} - y\tilde{x} = 0 \\ \tilde{x} \neq 0 \end{matrix} \right\}$$
$$\cong \{ (x, y, \tfrac{\tilde{y}}{\tilde{x}}) \mid y = x \tfrac{\tilde{y}}{\tilde{x}} \}$$
$$\cong \{ (x, \tfrac{\tilde{y}}{\tilde{x}}) \} = \mathrm{Spec}(k[x, \tfrac{\tilde{y}}{\tilde{x}}])$$

Similarly, $U_y = \mathrm{Spec}(k[y, \frac{\tilde{x}}{\tilde{y}}])$ and

$$U_x \cap U_y = \mathrm{Spec}(k[x, \tfrac{\tilde{y}}{\tilde{x}}, x^{-1}, (\tfrac{\tilde{y}}{\tilde{x}})^{-1}]) \qquad x \cdot \tfrac{\tilde{y}}{\tilde{x}} \qquad (\tfrac{\tilde{y}}{\tilde{x}})^{-1}$$
$$\cong \downarrow \qquad\qquad\qquad\qquad \uparrow \qquad\qquad \uparrow$$
$$U_y \cap U_x = \mathrm{Spec}(k[y, \tfrac{\tilde{x}}{\tilde{y}}, y^{-1}, (\tfrac{\tilde{x}}{\tilde{y}})^{-1}]) \qquad y \qquad\qquad \tfrac{\tilde{x}}{\tilde{y}}$$

Def The exceptional divisor of $Bl_Z Y$ is

$$E_Z(Y) = \beta^{-1}(Z) = \mathrm{Proj}_Z\Big(\bigoplus_{n\geq 0} \underbrace{\mathcal{I}_Z^n/\mathcal{I}_Z^{n+1}}_{= i_Z^* \mathcal{I}_Z}\Big)$$

Example:

$$E_0(\mathbb{A}^2) = \mathrm{Proj}_{\cancel{Z} k}\big(k[x,y][\tilde{x},\tilde{y}]/(x\tilde{y}-y\tilde{x}, x, y)\big).$$

$$= \mathrm{Proj}_k(k[\tilde{x},\tilde{y}]) = \mathbb{P}^1_{\tilde{x},\tilde{y}}$$

$$= \{(0,0)[\tilde{x}:\tilde{y}]\} \subseteq \{(x,y)[\tilde{x}:\tilde{y}] \mid x\tilde{y}-y\tilde{x}=0\}$$

$$= Bl_0(\mathbb{A}^2)$$

$$E_0(\mathbb{A}^2) \cap U_x = \mathrm{Spec}\left(k[x, \tfrac{\tilde{y}}{\tilde{x}}]/(x)\right) \cong \mathrm{Spec}(k[\tfrac{\tilde{y}}{\tilde{x}}])$$

$$E_0(\mathbb{A}^2) \cap U_y = \mathrm{Spec}\left(k[y, \tfrac{\tilde{x}}{\tilde{y}}]/(y)\right) \cong \mathrm{Spec}(k[\tfrac{\tilde{x}}{\tilde{y}}])$$

$\Rightarrow E_0(\mathbb{A}^2) \subset Bl_0(\mathbb{A}^2)$ effective Cartier divisor.

Theorem (Universal property)

The blow up of Y at Z is final amongst all morphisms $\pi: W \to Y$ such that $\pi^{-1}(Z)$ is an effective Cartier divisor.

Proof that $Bl_0\mathbb{A}^2 \to \mathbb{A}^2$ satisfies the UP

We saw that $E_0(\mathbb{A}^2)$ is an effective Cartier divisor. Let $\pi : W \to \mathbb{A}^2$ with $\pi^{-1}(0)$ Cartier.

Step 1: $W = \operatorname{Spec}(R)$ R local ring.

$R \supseteq (\pi^\flat(x), \pi^\flat(y))$ principal.

NAK $\Rightarrow$ $\pi^\flat(x)$ or $\pi^\flat(y)$ generates, say $\pi^\flat(x)$

Write $\pi^\flat(y) = f\,\pi^\flat(x)$

get ring morphism

$$k[x, \tfrac{\tilde{y}}{x}] \longrightarrow R \qquad x \mapsto \pi^\flat(x),\ \tfrac{\tilde{y}}{x} \mapsto f$$

$$\rightsquigarrow \begin{tikzcd} \operatorname{Spec}(R) \arrow[r] \arrow[dr] & U_x \arrow[r] & Bl_0(\mathbb{A}^2) \arrow[dl] \\ & \mathbb{A}^2 & \end{tikzcd}$$

Step 2: General case.

Step 1 $\Rightarrow$ $\exists$ open cover $W = \bigcup_i W_i$ and morphisms $\tilde{\pi}_i : W_i \to Bl_0\mathbb{A}^2$

Need to show they agree on overlaps.

We have $\tilde{\pi}_i|_{W_i \setminus \pi^{-1}(0)} = \pi|_{W_i \setminus \pi^{-1}(0)}$

$\Rightarrow$ $\tilde{\pi}_i|_{(W_i \cap W_j) \setminus \pi^{-1}(0)} = \tilde{\pi}_j|_{(W_i \cap W_j) \setminus \pi^{-1}(0)}$

$\pi^{-1}(0)$ Cartier $\Rightarrow$ $W \setminus \pi^{-1}(0)$ scheme-theoretically dense in W

$\Rightarrow$ $\tilde{\pi}_i|_{W_i \cap W_j} = \tilde{\pi}_j|_{W_i \cap W_j}$ $\square$

Facts

i) $\pi : Y' \to Y$ morphism

$$\begin{tikzcd} Bl_{\pi^{-1}(Z)}(Y') \arrow[r, dashed, "\exists!"] \arrow[d] & Bl_Z(Y) \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

ii) if π is flat, then Cartesian
(e.g. if π is open).

iii) $\beta|_{Bl \setminus E} : Bl_Z(Y) \setminus E_Z(Y) \longrightarrow Y \setminus Z$
is an isomorphism.

iv) $W \subseteq Y$ closed
$\Rightarrow Bl_{W\cap Z}(W) \hookrightarrow Bl_Z(Y)$ closed.
and $Bl_{W\cap Z}(W) = \overline{\beta^{-1}(W\setminus Z)}$
scheme theoretic closure.

Def $\tilde{W} = Bl_{W\cap Z}(W) = \overline{\beta^{-1}(W \setminus Z)}$ is called the strict transform of W in $Bl_Z(Y)$

v) if $Z \subset Y$ is regular, i.e. locally cut out by a regular sequence, then

$$\bigoplus_{n \geq 0} \mathcal{I}_Z^n \cong Sym_Y^\circ(\mathcal{I}_Z)$$

vi) $Y = Spec(A)$, $Z = V(f_1, \dots, f_n)$

$\alpha_{f_1,\dots,f_n} : U = Y \setminus Z \longrightarrow \mathbb{P}^{n-1} \times Spec(A) = \mathbb{P}^{n-1}_A$

given by $A^n \xrightarrow{(f_1,\dots,f_n)} A$,

$Bl_Z(Y) = \overline{graph(\alpha_{f_1,\dots,f_n})} \subset \mathbb{P}^{n-1}_A$

Example $L = V(y - mx) \subseteq \mathbb{A}^2$
$C = V(y^2 - x^2(x+1)) \subseteq \mathbb{A}^2$.

Compute their strict transforms in $Bl_0(\mathbb{A}^2)$ on $\tilde{x} \neq 0$ patch U_x:

$U_x \supseteq \beta^{-1}(L) \cap U_x$ cut out by equation

$$k[x, \tfrac{\tilde y}{\tilde x}] \ni \pi^\flat(y - mx) = x\frac{\tilde y}{\tilde x} - mx = x\left(\frac{\tilde y}{\tilde x} - m\right)$$

on $\beta^{-1}(L \setminus 0) \cap U_x$ x is invertible $(E \cap U_x = V(x))$

$\Rightarrow \beta^{-1}(L \setminus 0) \cap U_x \subseteq U_x$ cut out by $\frac{\tilde y}{\tilde x} - m$

$\Rightarrow \overline{\beta^{-1}(L \setminus 0) \cap U_x} \subseteq U_x$ cut out by —"—

$\forall U_y \Rightarrow \overline{\beta^{-1}(L \setminus 0)} = \tilde L = V(\tilde y - m\tilde x)$

$\beta^{-1}(C) \cap U_x$ cut out by equation.

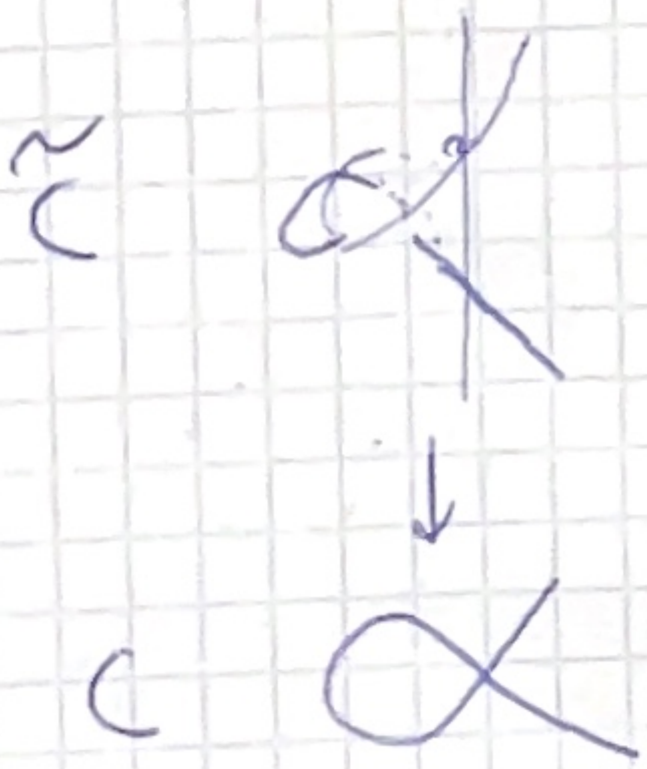
$$k[x, \tfrac{\tilde y}{\tilde x}] \ni \pi^\flat(y^2 - x^2(x+1)) = x^2\left(\frac{\tilde y}{\tilde x}\right)^2 - x^2(x+1)$$
$$= x^2\left((\tilde y/\tilde x)^2 - (x+1)\right)$$

$\Rightarrow \beta^{-1}(C \setminus 0) \cap U_x$ cut out by $\left(\frac{\tilde y}{\tilde x}\right)^2 - (x+1)$

$\Rightarrow \overline{\beta^{-1}(C \setminus 0)} = V(\tilde y^2 - \tilde x^2(x+1))$

$(\tilde L \cap E) \cap U_x = V\left(\frac{\tilde y}{\tilde x} - m, x\right) = \{(0, m)\}$

$(\tilde C \cap E) \cap U_x = V\left(\left(\frac{\tilde y}{\tilde x}\right)^2 - (x+1), x\right) = V\left(\left(\frac{\tilde y}{\tilde x}\right)^2 - 1, x\right)$
$= \{(0, \pm 1)\}$



Blowing up a point on a surface

X surface, $p \in X$.

$\beta: \tilde{X} = Bl_p(X) \longrightarrow X$, $E = E_p(X)$

<u>Lemma</u> $C \subset X$ irred., $\tilde{C} \subset \tilde{X}$ strict transform

$\beta^* C = \tilde{C} + \mathrm{mult}_p(C) E$

<u>Proof</u> $x, y \in \mathcal{O}_{X,p}$ local parameters

$f \in \mathcal{O}_{X,p}$ equation cutting out C at p.

We have an isomorphism of graded $\mathcal{O}_{X,p}$-algebras.

$$\mathcal{O}_{X,p}[\tilde{x}, \tilde{y}] / (x\tilde{y} - y\tilde{x}) \xrightarrow{\sim} \bigoplus_{n \geq 0} \mathfrak{m}_{X,p}^n$$

$r := \mathrm{mult}_p(C) = \max \{ r \mid f \in \mathfrak{m}_{X,p}^r \}$

$\mathfrak{m}^n / \mathfrak{m}^{n+1}$ v.s. with k-basis $x^n, x^{n-1}y, \dots, xy^{n-1}, y^n$

X ns $\Rightarrow$ can write $f = f_r(x,y) + g$

for $f_r(\tilde{x}, \tilde{y}) \in k[\tilde{x}, \tilde{y}]$, $g \in \mathfrak{m}_{X,p}^{r+1}$

$\tilde{x} \neq 0$ $\beta^b(f) = f_r(x, x\frac{\tilde{y}}{\tilde{x}}) + \beta^b(g)$

$= x^r \left(f_r(1, \frac{\tilde{y}}{\tilde{x}}) + \beta^b(g)/x^r \right)$

on $\{\tilde{x} \neq 0\}$ E given by $x = 0$

$\Rightarrow \mathrm{ord}_E(\beta^* C) = r = \mathrm{mult}_p(C)$ $\square$

Proposition

$$\mathrm{Pic}(X) \oplus \mathbb{Z} \longrightarrow \mathrm{Pic}(\tilde{X})$$
$$(D, n) \longmapsto \beta^* D + nE$$

is an isomorphism.

For all $D_1, D_2 \in \mathrm{Pic}(X)$ have

$$(D_1 . D_2) = (\beta^* D_1 . \beta^* D_2)$$
$$(\beta^* D_1 . E) = 0$$
$$E^2 = -1$$

Proof Bertini $\Rightarrow$ can replace D_1, D_2 by linearly equivalent divisors which don't pass through p.

$\Rightarrow (\beta^* D_1 . \beta^* D_2) = (D_1 . D_2)$, $(\beta^* D_1 . E) = 0$

Choose irred curve C passing through p w/ $\mathrm{multip}(C) = 1$ (exists b/c $p \in X$ codim 2)

$(\tilde{C} . E) = 1$, $(\beta^* C . E) = 0$

Lemma $\Rightarrow$ $E^2 = -1$

Alternatively, $\mathcal{O}_{\tilde{X}}(+E)|_E \cong \mathcal{O}_{\mathbb{P}^1}(-1)$

$\Rightarrow E^2 = \deg_E(\mathcal{O}_{\tilde{X}}(E)|_E) = -1$ can check gluing maps.

have exact sequence

$$0 \to \mathbb{Z} \to \mathrm{Pic}(\tilde{X}) \to \mathrm{Pic}(\tilde{X} \setminus E) \to 0$$
$$n \longmapsto nE$$
$$\mathrm{Pic}(\tilde{X} \setminus E) = \mathrm{Pic}(X \setminus p) \cong \mathrm{Pic}(X)$$

because $(nE)^2 = -n^2$

"extend over codim 2 locus"

$\beta^* : \mathrm{Pic}(X) \to \mathrm{Pic}(\tilde{X})$ is a section.

$\square$